A MATRIX-FREE LOCALLY-IMPLICIT SCHEME FOR NAVIER-STOKES EQUATIONS

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ABSTRACT

A locally-implicit scheme for steady-state solution of the thin-layer Navier-Stokes Equations is simplified by elimination of coefficient matrices. This reduces both arithmetic computation and computer storage requirements. An added benefit is the simplification of the algorithm which eases the coding task. The locally-implicit scheme uses finite-volume spatial discretization, locally-implicit time integration, Jameson-type artificial dissipation terms and a modified Gauss-Seidel iteration. The modified method is tested for subsonic and transonic flows over an RAE 2822 airfoil.

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LIST OF SYMBOLS AND ABBREVIATIONS

a	see Equation (2), also speed of sound
A, B	Jacobian matrices corresponding to inviscid flux vectors
c	see Equation (6)
CFL	Courant number
C_p	pressure coefficient
d	artificial dissipation
du	correction quantity for Δu_j [see Equation(8)]
e	energy
E, F	inviscid flux vectors
g	amplification factor
i	$\sqrt{-1}$
I	identity matrix
J	Jacobian of metric tensor for coordinate transformation
L	any operator
M	Mach number
p	pressure
Pr	Prandtl number
Q	vector of dependent, conservation variables
$\Delta Q, \Delta u$	change in the solution per time step
r	see Equation(6)
R	simplified viscous flux Jacobian (diagonal) matrix
Re	Reynolds number
Res	residual
RHS	right hand side

$\widehat{S}$	viscous flux vector
t	time
u	general one-dimensional variable, Cartesian velocity component in x direction
v	Cartesian velocity component in y direction
α	see Equation (20), also angle of attack (see Table 1)
γ	ratio of specific heats
$\epsilon^{(2)}$	factor used in artificial dissipation
$\epsilon^{(4)}$	factor used in artificial dissipation
ζ	see Equation (20)
$\kappa^{(2)}$	second-difference dissipation constant
$\kappa^{(4)}$	fourth-difference dissipation constant
μ	dynamic viscosity
ν	factor used in artificial dissipation, also kinematic viscosity
ρ	density
τ	time
ω	relaxation factor

Subscripts

j	x-direction
k	y-direction
ξ, η	curvilinear coordinates
∞	free stream condition

Superscripts

m	iteration sweep count
n	time level

CHAPTER 1

INTRODUCTION

Though computational fluid dynamics (CFD) has been in existence for less than three decades, it is already recognized as an increasingly powerful tool for aerodynamic design of aerospace vehicles[1]. Flight test, ground test, and CFD compliment one another as aerospace vehicle design requirements become more demanding. Besides being used directly in the design process, CFD also plays an important role in furthering scientific understanding of complex flow phenomena.

Within the CFD spectrum are various levels of maturity. At one end of the spectrum are panel codes that solve linear-type flows for which CFD is quite mature. These codes can readily handle very complicated geometries in reasonable computer run times. At the other end of the spectrum are the Euler and Navier-Stokes codes which handle the complex physics of nonlinear-type flow. These codes currently require substantial computer resources (both memory and time) and are limited to simple geometries (relative to linear-type flow solvers), but provide essential physical insight into many classes of flow problems. To make Euler and Navier-Stokes codes even more useful than they currently are, gains must be made in several areas such as improvement in speed of convergence; improvement in handling complex geometry; reduction in numerical diffusion for flows containing shocks, wakes and vortex structures; and better turbulence modeling[1]. The scheme discussed in this thesis addresses the area of increasing computational efficiency of algorithms.

Current useful Euler and Navier-Stokes schemes fall into one of two categories: explicit or implicit methods. MacCormack[2], in comparing these methods, points out the need to use an implicit procedure to avoid restricting CFL

numbers to small values. He states that Newton iteration and line Gauss-Seidel procedures show potential for significantly increasing numerical efficiency. Chakravarthy[3] has shown various relaxation procedures for implicit schemes. Reddy and Jacocks[4] have presented a locally-implicit scheme for solving the Euler equations for large problems using a modified Gauss-Seidel relaxation procedure. They find a CFL number of about ten to be appropriate for their scheme. Reddy and Nayani[5], and Nayani[6] have extended the locally-implicit scheme to solution of the two-dimensional Navier-Stokes equations. The work presented in this thesis is a modification to that scheme. The modification results in elimination of seven four-by-four matrices thereby making the scheme matrix-free. The modified scheme involves less arithmetic computation and requires less computer storage. Primary motivation for the work presented in this thesis is to establish the viability of the matrix-free scheme.

The scheme Nayani presented in his dissertation[6] is referred to throughout this thesis as the "original" scheme. The modification reported in this thesis is referred to as the "modified" scheme. Nayani's computer code was used as the starting point for the results presented in this thesis. Parametric studies of various relaxation parameters with regard to the convergence process and multigrid acceleration of convergence of the scheme are not addressed here, since such features would be similar to the original scheme.

CHAPTER 2

LOCALLY-IMPLICIT SCHEME FOR ONE-DIMENSIONAL MODEL EQUATION

In this chapter, a model equation is used to describe the locally-implicit method. It starts with a finite difference scheme based on central differences for spatial derivatives and Euler implicit time integration. A modified symmetric Gauss-Seidel iteration is developed for solving the implicit equations at every time step. The scheme is shown to be unconditionally stable by Fourier stability analysis.

2.1 Model Equations

Nayani[6] has demonstrated the basic locally-implicit scheme and its stability with the one-dimensional diffusion equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Reddy and Nayani[5] have established the unconditional stability of the locally-implicit scheme for the linearized Burgers equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{2}$$

which serves as a model for the Navier-Stokes equations. The coefficient a represents a convection velocity which is independent of x and can be positive or negative. The coefficient ν represents a generalized diffusion coefficient corresponding to the fluid viscosity or an artificial dissipation coefficient designed to be effective near shocks ($\nu \approx \kappa^{(2)}|a|\Delta x$, where $\kappa^{(2)}$ is a second-difference dissipation constant obtained through numerical experimentation).

The following one-dimensional model equation will be used in this thesis to present the modified locally-implicit scheme and its stability.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\nu\frac{\partial^4 u}{\partial x^4} \tag{3}$$

In Equation (3), ν represents an artificial dissipation coefficient designed to be effective everywhere in the flow, except near shocks ($\nu \approx \kappa^{(4)}|a|\Delta x^3$, where $\kappa^{(4)}$ is a fourth-difference dissipation constant). Artificial dissipation of this type has been shown by Jameson[7] and others to suppress the non-linear instabilities which arise in central difference schemes for convection dominated flows.

2.2 Euler Implicit Scheme for Model Equation

Reddy and Nayani[5] have analyzed the locally-implicit scheme for Equation (2) with central difference approximations for the spatial derivatives and Euler implicit time integration.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = \nu\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \tag{4}$$

The same basic scheme will be demonstrated for Equation (3). As in Equation (4), the Euler implicit time integration and central spatial difference approximations for the spatial derivatives are used.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} = -\nu\frac{u_{j+2}^{n+1} - 4u_{j+1}^{n+1} + 6u_j^{n+1} - 4u_{j-1}^{n+1} + u_{j-2}^{n+1}}{\Delta x^4} \tag{5}$$

A direct solution of Equation (5) along with its accompanying boundary conditions requires solution of a pentadiagonal system of equations for each time step. In multidimensions, the matrix is too big for a direct-solution method.

Therefore, the locally-implicit method described by Reddy and Nayani[5] is used to obtain an asymptotic steady-state solution of Equation (3).

The delta form of Equation (5) is sought by rewriting Equation (5) with the use of the following definitions for Δu, c and r.

$$\Delta u_j \equiv u_j^{n+1} - u_j^n, \quad c \equiv \frac{a\Delta t}{\Delta x} = CFL, \quad r \equiv \frac{\nu \Delta t}{\Delta x^4} = \frac{\kappa^{(4)}|a|\Delta t}{\Delta x} = \kappa^{(4)}|c|$$

$$\begin{aligned}\Delta u_j &+ \frac{c}{2}(\Delta u_{j+1} - \Delta u_{j-1}) \\ &+ r(\Delta u_{j+2} - 4\Delta u_{j+1} + 6\Delta u_j - 4\Delta u_{j-1} + \Delta u_{j-2}) = Res_j^n\end{aligned} \tag{6}$$

$$Res_j^n = -\frac{c}{2}(u_{j+1}^n - u_{j-1}^n) - r(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n) \tag{7}$$

Res_j^n is the residual. A steady-state solution is obtained for Equation (5) by driving the residual to zero.

In the standard Gauss-Seidel iteration procedure, when sweeping left to right (j increasing), Δu_j is solved for by setting Δu_{j+1} to zero and using the value for Δu_{j-1} available from the last iteration step. This procedure, however, is not stable because of the convection terms, even if a symmetric Gauss-Seidel scheme is used. Therefore, a modified Gauss-Seidel procedure[4] that is unconditionally stable is used which employs an inner iteration. First, denote

$$du_j \equiv \Delta u_j^{(m+1)} - \Delta u_j^{(m)}$$

where m is the inner iteration index count and du_j is the correction quantity for Δu_j in the m^{th} inner iteration sweep. Substituting this expression for du_j into Equation (6) gives

$$du_j + \frac{c}{2}(du_{j+1} - du_{j-1})$$
$$+ r(du_{j+2} - 4du_{j+1} + 6du_j - 4du_{j-1} + du_{j-2}) = RHS \tag{8}$$

$$RHS = Res_j^n - [\Delta u_j^{(m)} + \frac{c}{2}(\Delta u_{j+1}^{(m)} - \Delta u_{j-1}^{(m)})$$
$$+ r(\Delta u_{j+2}^{(m)} - 4\Delta u_{j+1}^{(m)} + 6\Delta u_j^{(m)} - 4\Delta u_{j-1}^{(m)} + \Delta u_{j-2}^{(m)})]$$

For a left-to-right sweep, the newly computed du's (i.e., du_{j-1} and du_{j-2}) are brought over to the right-hand side of Equation (8) producing the following equation.

$$du_j + \frac{c}{2}(du_{j+1}) + r(du_{j+2} - 4du_{j+1} + 6du_j) = RHS \tag{9}$$

$$RHS = Res_j^n - L_j(\Delta u)$$

$$L_j(\Delta u) = \Delta u_j^{(m)} + \frac{c}{2}(\Delta u_{j+1}^{(m)} - \Delta u_{j-1}^{(m+1)})$$
$$+ r(\Delta u_{j+2}^{(m)} - 4\Delta u_{j+1}^{(m)} + 6\Delta u_j^{(m)} - 4\Delta u_{j-1}^{(m+1)} + \Delta u_{j-2}^{(m+1)})$$

So far, this development is identical to that shown by Nayani[6] for the heat equation and is the departure point for the present scheme.

The latest available values for u_j are denoted by starred quantities as follows.

$$u_j^* \equiv u_j^n + \Delta u_j^{(m)}, \quad u_{j+1}^* \equiv u_{j+1}^n + \Delta u_{j+1}^{(m)}, \quad u_{j-1}^* \equiv u_{j-1}^n + \Delta u_{j-1}^{(m+1)} \tag{10}$$

u_{j-2}^* and u_{j+2}^* have similar representations. Similarly, in the right-to-left sweep, starred quantities represent the latest available values also. The terms in the residual [Equation (7)] can be combined with all the other terms in RHS except $\Delta u_j^{(m)}$, to produce

$$RHS = -\Delta u_j^{(m)} - \frac{c}{2}(u_{j+1}^* - u_{j-1}^*)$$
$$- r(u_{j+2}^* - 4u_{j+1}^* + 6u_j^* - 4u_{j-1}^* + u_{j-2}^*) \quad (11)$$

or

$$RHS = Res_j^* - \Delta u_j^{(m)} \quad (12)$$

$$Res_j^* = -\frac{c}{2}(u_{j+1}^* - u_{j-1}^*) - r(u_{j+2}^* - 4u_{j+1}^* + 6u_j^* - 4u_{j-1}^* + u_{j-2}^*) \quad (13)$$

where Res_j^* is the residual computed for the latest values of u.

The left-to-right scheme is given as

$$du_j + \frac{c}{2}(du_{j+1}) + r(du_{j+2} - 4du_{j+1} + 6du_j) = Res_j^* - \Delta u_j^{(m)} \quad (14)$$

If the right-hand side is driven to zero, a time-accurate solution to the discrete unsteady problem [Equation (5)] for u_j^{n+1} results. The left side of Equation (14) can be modified to achieve efficiency and convergence of the scheme without altering the actual solution.

First, approximate du_{j+2} and du_{j+1} by du_j. This gives

$$(1 + \frac{c}{2} + 3r)du_j = Res_j^* - \Delta u_j^{(m)} \quad (15)$$

Using this approximation, for a right-to-left sweep, Equation (14) is

$$(1 - \frac{c}{2} + 3r)du_j = Res_j^* - \Delta u_j^{(m)} \quad (16)$$

This type of symmetric Gauss-Seidel iteration is not stable. The symmetric Gauss-Seidel iteration is modified by replacing c by $|c|$ and using

$$(1 + \frac{|c|}{2} + 3r)$$

as the coefficient of du_j for both left-to-right and right-to-left sweeps.

For initialization of the inner iteration, set $\Delta u_j^{(0)} \equiv 0$. Thus, the modified scheme is represented by

$$(1 + \frac{|c|}{2} + 3r)du_j = Res_j^* - \Delta u_j^{(m)} \tag{17}$$

None of the modifications have jeopardized the ability to obtain time-accurate solutions to Equation (5). For time accuracy, symmetric sweeps per time step are performed until the right-hand side of Equation (17) becomes sufficiently close to zero. A symmetric sweep for this one-dimensional case is one left-to-right sweep followed by one right-to-left sweep. In this thesis, since the steady-state solution is sought, only one or two symmetric sweeps per time step are necessary while advancing the solution in time until desired convergence is obtained (i.e., until Res_j^* is driven to zero).

The major advantage of this modification to the scheme is the resulting reduction in both, arithmetic computation and computer storage requirements. This fact will become evident in Section 3.2.

2.3 Local Stability Analysis for Model Equation

Local linear stability analyses for this locally-implicit scheme have already been presented for two of the three model equations of interest. Both, Burgers' equation[5] and the heat diffusion equation[6] have been shown to be unconditionally stable in a local linear sense for all CFL numbers. The stability analysis for the third model equation of interest [Equation (3)] is presented in this section.

The standard Fourier stability analysis is performed on Equation (17) after expanding Res_j^* from its $*$ notation back to the pure delta form.

$$(1 + \frac{|c|}{2} + 3r)du_j = RHS \tag{18}$$

$$RHS = RHS_{L-R} = Res_j^n - [\Delta u_j^{(m)} + \frac{c}{2}(\Delta u_{j+1}^{(m)} - \Delta u_{j-1}^{(m+1)}) + r(\Delta u_{j+2}^{(m)} - 4\Delta u_{j+1}^{(m)} + 6\Delta u_j^{(m)} - 4\Delta u_{j-1}^{(m+1)} + \Delta u_{j-2}^{(m+1)})]$$

or

$$RHS = RHS_{R-L} = Res_j^n - [\Delta u_j^{(m)} + \frac{c}{2}(\Delta u_{j+1}^{(m+1)} - \Delta u_{j-1}^{(m)}) + r(\Delta u_{j+2}^{(m+1)} - 4\Delta u_{j+1}^{(m+1)} + 6\Delta u_j^{(m)} - 4\Delta u_{j-1}^{(m)} + \Delta u_{j-2}^{(m)})]$$

$$Res_j^n = -[\frac{c}{2}(u_{j+1}^n - u_{j-1}^n) + r(u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n)]$$

$$du_j = \Delta u_j^{(m+1)} - \Delta u_j^{(m)}$$

One symmetric sweep will be considered in this analysis. For the left-to-right sweep, $m = 0$. Recalling that $\Delta u_j^{(0)}$ is set to zero, Equation (18) becomes

$$(1 + \frac{|c|}{2} + 3r)du_j = RHS_{L-R} \tag{19}$$

$$RHS_{L-R} = Res_j^n - [\frac{c}{2}(-\Delta u_{j-1}^{(1)}) + r(-4\Delta u_{j-1}^{(1)} + \Delta u_{j-2}^{(1)})]$$

$$du_j = \Delta u_j^{(1)} - \Delta u_j^{(0)} = \Delta u_j^{(1)}$$

$$u_j^{\bar{n}} = u_j^n + \Delta u_j^{(1)}$$

$$u_j^{n+1} = u_j^n + \Delta u_j^{(2)}$$

Discrete modal solution of Equation (18) is sought in the form

$$u_j^n = V^n e^{ij\alpha\Delta x} = V^n e^{ij\zeta}, \quad i = \sqrt{-1}, \quad \zeta = \alpha\Delta x, \quad 0 \le \zeta \le \pi \tag{20}$$

$$u_j^{\bar{n}} = V^{\bar{n}} e^{ij\zeta} \tag{21}$$

$$\Delta u_j^{(1)} = u_j^{\bar{n}} - u_j^n = (V^{\bar{n}} - V^n)e^{ij\zeta} \tag{22}$$

$$\Delta u_j^{(2)} = u_j^{n+1} - u_j^n = (V^{n+1} - V^n)e^{ij\zeta} \tag{23}$$

Substituting Equations (20)-(22) into Equation (19) and dividing each side of the resulting equation by $e^{ij\zeta}$ and then simplifying gives

$$T_1(V^{\bar{n}} - V^n) = T_2 V^n \tag{24}$$

$$b = 1 + \frac{|c|}{2} + 3r$$

$$T_1 = b - [\frac{c}{2} + r(4 - e^{-i\zeta})]e^{-i\zeta}$$

$$T_2 = -icSin\zeta - r[2Cos(2\zeta) - 8Cos\zeta + 6]$$

The amplification factor g for the scheme is defined by the equation

$$V^{n+1} = gV^n \tag{25}$$

So, for the left-to-right sweep the amplification factor $\bar{g}$ is

$$\bar{g} = \frac{V^{\bar{n}}}{V^n} = 1 + \frac{T_2}{T_1} \tag{26}$$

For the right-to-left sweep, Equation (18) becomes

$$(1 + \frac{|c|}{2} + 3r)du_j = RHS_{R-L} \tag{27}$$

$$RHS_{R-L} = Res_j^n - [\Delta u_j^{(1)} + \frac{c}{2}(\Delta u_{j+1}^{(2)} - \Delta u_{j-1}^{(1)})$$
$$+ r(\Delta u_{j+2}^{(2)} - 4\Delta u_{j+1}^{(2)} + 6\Delta u_j^{(1)} - 4\Delta u_{j-1}^{(1)} + \Delta u_{j-2}^{(1)})]$$
$$du_j = \Delta u_j^{(2)} - \Delta u_j^{(1)}$$

Equations (20)-(23) are substituted into Equation (27) and each side of the resulting equation is divided by $e^{ij\zeta}$ and then simplified giving

$$T_3(V^{n+1} - V^n) = T_2 V^n + T_4(V^{\bar{n}} - V^n) \tag{28}$$
$$T_3 = b + [\frac{c}{2} - r(4 - e^{i\zeta})]e^{i\zeta}$$
$$T_4 = \frac{|c|}{2} - 3r + [\frac{c}{2} + r(4 - e^{-i\zeta})]e^{-i\zeta}$$

After the right-to-left sweep is performed the amplification factor for the scheme is

$$g = \frac{V^{n+1}}{V^n} = 1 + \frac{T_2}{T_3} + \frac{T_4}{T_3}(\bar{g} - 1) \tag{29}$$

Substituting Equation (26) into Equation (29) and simplifying gives the scheme amplification factor as

$$g = 1 + \frac{T_2}{T_3}(1 + \frac{T_4}{T_1}) \tag{30}$$

The stability requirement is

$$|g| \leq 1 \tag{31}$$

Therefore, to have a stable scheme, the following must be true.

$$|g| = |1 + \frac{T_2}{T_3}(1 + \frac{T_4}{T_1})| \leq 1 \tag{32}$$

Figure 1 shows a plot of the modulus of the amplification factor versus the phase angle for Equation (32). CFL varies from one to 100 and $\nu = \frac{1}{32}$. As stated at the beginning of this section, the locally-implicit scheme is indeed stable for the one-dimensional model equation which models the fourth-order numerical dissipation. From the other stability analyses that have been presented by Reddy and Nayani[5], and Nayani[6], a CFL number of 10 was chosen. This is a good choice for this model equation as well.

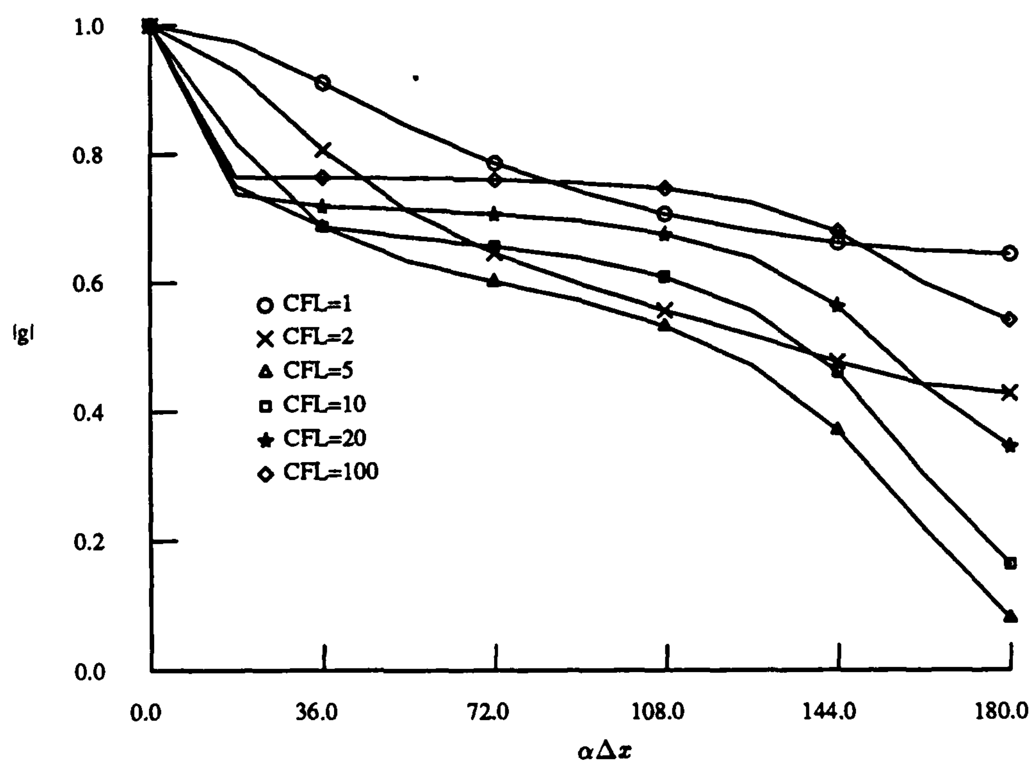


Figure 1. Stability Plot of Modified Locally-Implicit Scheme Applied to 1-D Equation (3), ($\nu = \frac{1}{32}$)

CHAPTER 3

LOCALLY-IMPLICIT SCHEME FOR NAVIER-STOKES EQUATIONS

Nayani gives a detailed derivation of the original scheme in his dissertation[6]. Relevant equations of the scheme are presented in Section 3.1 without rederivation. The technique he has presented possesses the following salient features: finite volume spatial descretization, Jameson-type artificial dissipation terms[7], modified Gauss-Seidel inner iteration. For the steady-state solution, he has used local time integration which varies the time step cell-by-cell in order to maintain a constant CFL number throughout the computational grid. Multigrid technique has been used for acceleration of convergence, but that aspect of the code is not addressed in this thesis. He has applied the scheme to the Navier-Stokes Equations incorporating the thin-layer approximation and using the Baldwin-Lomax[8] turbulence model. The modification to the original scheme (presented in section 3.2) does not change any of the above features.

3.1 Original Locally-Implicit Scheme for Navier-Stokes Equations

Nayani[6] starts his development with the Navier-Stokes Equations in nondimensional form for generalized curvilinear coordinates. Then he makes the thin-layer approximation and simplifies the equations to arrive at

$$\begin{aligned}\frac{\partial}{\partial \tau}(J^{-1}Q) + \frac{\partial}{\partial \xi}(y_\eta E - x_\eta F) \\ + \frac{\partial}{\partial \eta}(-y_\xi E + x_\xi F) - Re^{-1}\frac{\partial}{\partial \eta}\widehat{S} = 0\end{aligned} \tag{33}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(e+p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(e+p) \end{bmatrix}$$

$$\widehat{S} = J^{-1} \begin{bmatrix} 0 \\ (\eta_x m_1 + \eta_y m_2) \\ (\eta_x m_2 + \eta_y m_3) \\ \eta_x(um_1 + vm_2 + m_4) + \eta_y(um_2 + vm_3 + m_5) \end{bmatrix}$$

$$\begin{aligned}
m_1 &= \frac{\mu}{3}(4\eta_x u_\eta - 2\eta_y v_\eta) \\
m_2 &= \mu(\eta_y u_\eta + \eta_x v_\eta) \\
m_3 &= \frac{\mu}{3}(-2\eta_x u_\eta + 4\eta_y v_\eta) \\
m_4 &= \frac{\mu}{Pr(\gamma-1)}\eta_x \partial_\eta (a^2) \\
m_5 &= \frac{\mu}{Pr(\gamma-1)}\eta_y \partial_\eta (a^2)
\end{aligned}$$

After integrating the Navier-Stokes Equations using a finite-volume approach, he derives the Euler implicit scheme in delta form using the modified Gauss-Seidel iteration technique.

$$Q^{n+1}_{j,k} = Q^n_{j,k} + \omega_{out}\Delta Q_{j,k} \tag{34}$$

ω_{out} is an outer iteration relaxation parameter. ΔQ is produced from the inner iteration

$$\Delta Q^{(m+1)}_{j,k} = \Delta Q^{(m)}_{j,k} + \omega_{in} dQ_{j,k}, \quad m = 0,1,2,3 \tag{35}$$

ω_{in} is an inner iteration relaxation parameter. m is the sweep number. Four sweeps (identified as $m = 0,1,2,3$) using Equation (35) make up one symmetric

sweep for this two-dimensional case. The $\Delta Q_{j,k}^{(4)}$ term is then used as $\Delta Q_{j,k}$ in Equation (34) which is added to $Q_{j,k}^{n}$ to give the new value $Q_{j,k}^{n+1}$.

The values for $dQ_{j,k}$ are produced from

$$C_{j,k} \cdot dQ_{j,k} = Res_{j,k}^{n} - L_{j,k}(\Delta Q) \tag{36}$$

$$\begin{aligned} Res_{j,k}^{n} = &-[y_\eta E^n - x_\eta F^n]\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k} - [-y_\xi E^n + x_\xi F^n]\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} \\ &+ Re^{-1}(\widehat{S}^n)\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} + d\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k} + d\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} \end{aligned} \tag{37}$$

Re is the Reynolds number, *d* is the artificial dissipation flux of Jameson[7] type given as follows.

$$\begin{aligned} d_{j+\frac{1}{2},k} = \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau} \Big[&\epsilon^{(2)}_{j+\frac{1}{2},k}(Q_{j+1,k} - Q_{j,k}) \\ &-\epsilon^{(4)}_{j+\frac{1}{2},k}(Q_{j+2,k} - 3Q_{j+1,k} + 3Q_{j,k} - Q_{j-1,k})\Big] \end{aligned}$$

$$\begin{aligned} \epsilon^{(2)}_{j+\frac{1}{2},k} &= \kappa^{(2)} \ \mathrm{Max}(\nu_{j+1,k} \ , \ \nu_{j,k}) \\ \epsilon^{(4)}_{j+\frac{1}{2},k} &= \mathrm{Max}[0, (\kappa^{(4)} - \epsilon^{(2)}_{j+\frac{1}{2},k})] \end{aligned}$$

where

$$\nu_{j,k} = \frac{|p_{j+1,k} - 2p_{j,k} + p_{j-1,k}|}{|p_{j+1,k} + 2p_{j,k} + p_{j-1,k}|}$$

d is defined similarly for other indices. $C_{j,k}$ is a diagonal matrix which is defined from a heuristic derivation (see Appendix 1).

$$\begin{aligned}
C_{j,k} = {} & CFL(\frac{J^{-1}_{j,k}}{\Delta\tau})(\frac{1}{2} + \frac{1}{CFL})I + \frac{1}{2}[\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k}I \\
& + 3\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k}I + 3\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}I \\
& + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}}I + 3\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I \\
& + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}}I + 3\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}I \\
& + Re^{-1}(R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}})]
\end{aligned} \tag{38}$$

where R is a diagonal matrix derived from the approximation of $\widehat{S}$ (see Reference 6). Note that Nayani's dissertation[6] contains a misprint for $C_{j,k}$ [his Equation (68)]. Equation (38) above reflects the correct $C_{j,k}$. The operator $L_{j,k}$ of Equation (36) is similar to L_j found in Equation (9) for the one-dimensional case.

$$\begin{aligned}
L_{j,k}(\Delta Q) = {} & \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}\Delta Q_{j-2,k} + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}\Delta Q_{j+2,k} \\
& + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}\Delta Q_{j,k-2} + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}\Delta Q_{j,k+2} \\
& + CL_{j,k}\Delta Q_{j-1,k} + CB_{j,k}\Delta Q_{j,k-1} + CC_{j,k}\Delta Q_{j,k} \\
& + CR_{j,k}\Delta Q_{j+1,k} + CT_{j,k}\Delta Q_{j,k+1}
\end{aligned} \tag{39}$$

where ΔQ's are the latest available values (i.e., some ΔQ's are at the sweep level (m) and some are at the level $(m+1)$, depending on the particular sweep direction at the time Equation (39) is being applied). The coefficient matrices $CC_{j,k}$, $CL_{j,k}$, $CR_{j,k}$, $CB_{j,k}$, and $CT_{j,k}$ are

$$
\begin{aligned}
CC_{j,k} = {} & \frac{1}{2}[(y_\eta A^n - x_\eta B^n)_{j+\frac{1}{2},k} - (y_\eta A^n - x_\eta B^n)_{j-\frac{1}{2},k}] \\
& + \frac{1}{2}[(-y_\xi A^n + x_\xi B^n)_{j,k+\frac{1}{2}} - (-y_\xi A^n + x_\xi B^n)_{j,k-\frac{1}{2}}] \\
& + Re^{-1}(R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}}) + \frac{J^{-1}_{j,k}}{\Delta\tau} I \\
& + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k} I + 3\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k} I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k} I \\
& + 3\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k} I + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}} I + 3\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}} I \\
& + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}} I + 3\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}} I
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
CL_{j,k} = {} & -\frac{1}{2}(y_\eta A^n - x_\eta B^n)_{j-\frac{1}{2},k} - \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k} I \\
& - \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k} I - 3\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k} I
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
CR_{j,k} = {} & \frac{1}{2}(y_\eta A^n - x_\eta B^n)_{j+\frac{1}{2},k} - \frac{J^{-1}_{J+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k} I \\
& - 3\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k} I - \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k} I
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
CB_{j,k} = {} & -\frac{1}{2}(-y_\xi A^n + x_\xi B^n)_{j,k-\frac{1}{2}} - Re^{-1}R_{j,k-\frac{1}{2}} \\
& - \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}} I - \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}} I - 3\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}} I
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
CT_{j,k} = {} & \frac{1}{2}(-y_\xi A^n + x_\xi B^n)_{j,k+\frac{1}{2}} - Re^{-1}R_{j,k+\frac{1}{2}} \\
& - \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}} I - 3\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}} I - \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}} I
\end{aligned} \tag{44}
$$

where

$$A = \frac{\partial E}{\partial Q}, \quad B = \frac{\partial F}{\partial Q}$$

The above original scheme is designed to compute the steady-state solution to the thin-layer Navier-Stokes Equations (33). The solution is obtained when $Res^n_{j,k}$ [Equation (37)] is near enough to zero.

The $L_{j,k}$ operator reported by Nayani[6] is given by Equation (39). His code contains some additional smoothing factors that do not appear in his dissertation. The smoothing factors make the code more robust. For details, see Appendix 2.

3.2 Modification of Locally-Implicit Scheme for Navier-Stokes Equations

The modification to the original scheme eliminates the need for the coefficient matrices of Equations (40)-(44). By applying the procedure described in Section 2.2 for the one-dimensional equation to the two-dimensional Navier-Stokes equations, the resulting scheme becomes

$$C_{j,k} \cdot dQ_{j,k} = Res^*_{j,k} - \Delta Q^*_{j,k} \frac{J^{-1}_{j,k}}{\Delta \tau} \tag{45}$$

$$\Delta Q^*_{j,k} = Q^*_{j,k} - Q^n_{j,k} \tag{46}$$

$$\begin{aligned} Res^*_{j,k} = &-[y_\eta E^* - x_\eta F^*]\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k} - [-y_\xi E^* + x_\xi F^*]\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} \\ &+ Re^{-1}(\widehat{S}^*)\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} + d^*\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k} + d^*\Big|_{j,k-\frac{1}{2}}^{j,k+\frac{1}{2}} \end{aligned} \tag{47}$$

All but one of the terms in $L_{j,k}(\Delta Q)$ [Equation (39)] combine with the terms of $Res^n_{j,k}$ [Equation (37)] to produce $Res^*_{j,k}$, which, analogous to the one-dimensional scheme, is the residual computed from the most recent values of Q [some values are at the sweep level (m) and some are at the sweep level $(m+1)$].

The main advantage of this formulation of the scheme is the elimination of the four-by-four coefficient matrices $CC_{j,k}$, $CL_{j,k}$, $CR_{j,k}$, $CB_{j,k}$, $CT_{j,k}$ [Equations (40)-(44)] and the four-by-four flux Jacobian matrices A^n and B^n (note that A^n and B^n require significant computer memory since they are stored at every node). This eliminates all of the four-by-four matrices thereby making the scheme matrix free. These would be five-by-five matrices for the three-dimensional Navier-Stokes equations. For a two-dimensional flow problem of 128 x 32 cells, this modification reduces arithmetic computation and memory storage requirements by approximately 30 percent and 50 percent, respectively. The task of coding the scheme is simplified as well.

CHAPTER 4

RESULTS

Three of the test cases Nayani used have been computed using the modified scheme and are compared with the original scheme and experimental data[9]. The cases are shown in Table 1 below for flow over the RAE 2822 airfoil.

Table 1. Test Cases for Modified Scheme

CASE	M_∞	α(deg)	Re
2	0.676	2.40	5.7×10^6
4	0.725	2.92	6.5×10^6
5	0.730	3.19	2.7×10^6

The grid for the RAE 2822 airfoil is shown in Figure 2. It is exactly the same grid Nayani[6] used. An algebraic turbulence model (eddy viscosity model developed by Baldwin and Lomax[8]) has been used to obtain effective dynamic viscosity values at each cell. The results are shown in Figures 3-5.

Static pressure contours for the three cases using the modified scheme are shown in Figures 3a, 4a, and 5a. The pressure coefficient plots (Figures 3b, 4b, and 5b) show that the original and modified scheme give the same results. Case-by-case comparisons of convergence behavior for the original and modified schemes are shown in Figures 3c, 4c, and 5c. Those too are practically identical.

For all the computations, the CFL number has been varied over the first 250 iterations since impulsive-start boundary conditions on the airfoil have been

used. For the first 150 iterations, the CFL number is set to one. For the next 100 iterations, it is set to five. For the remaining iterations, the CFL number is 10. There are alternate ways of starting the solution process, such as gradual implementation of boundary conditions over a number of time steps instead of impulsive-start, but such techniques are not demonstrated here.

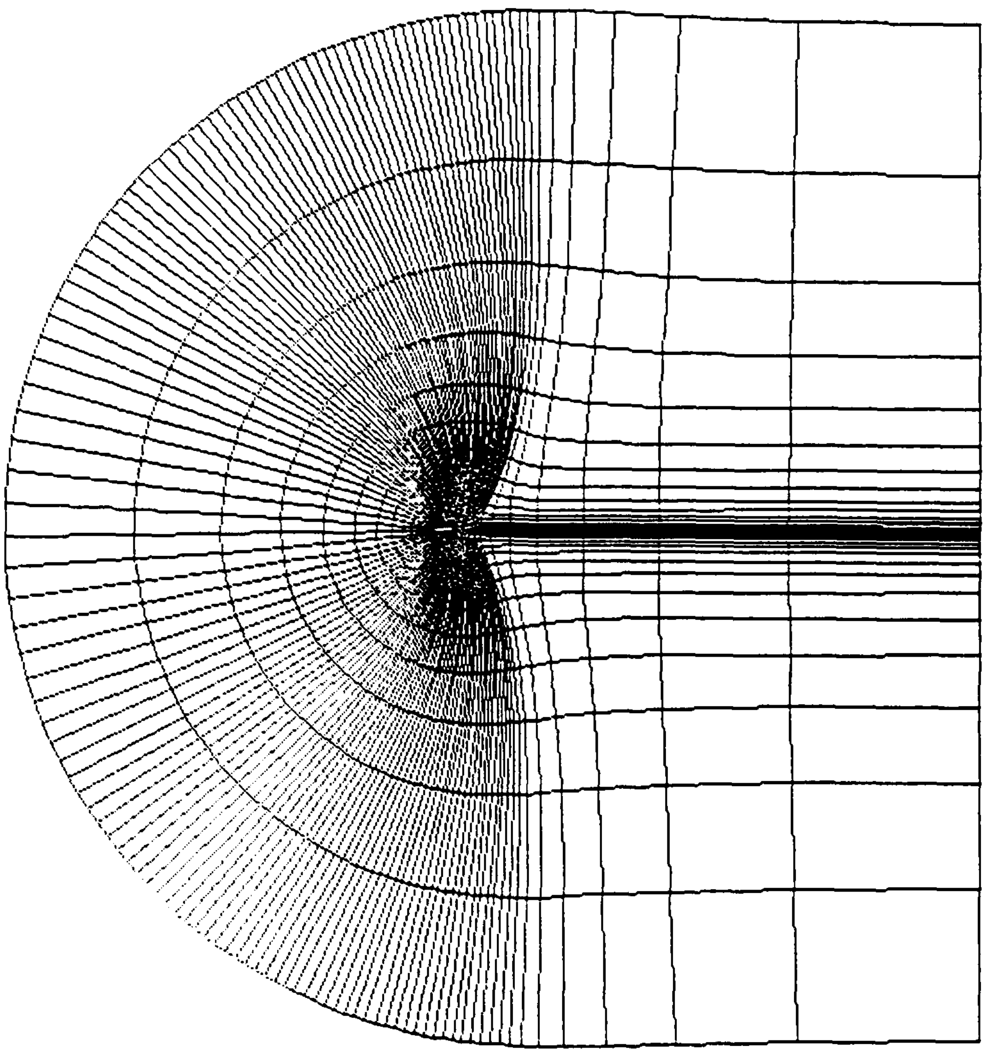
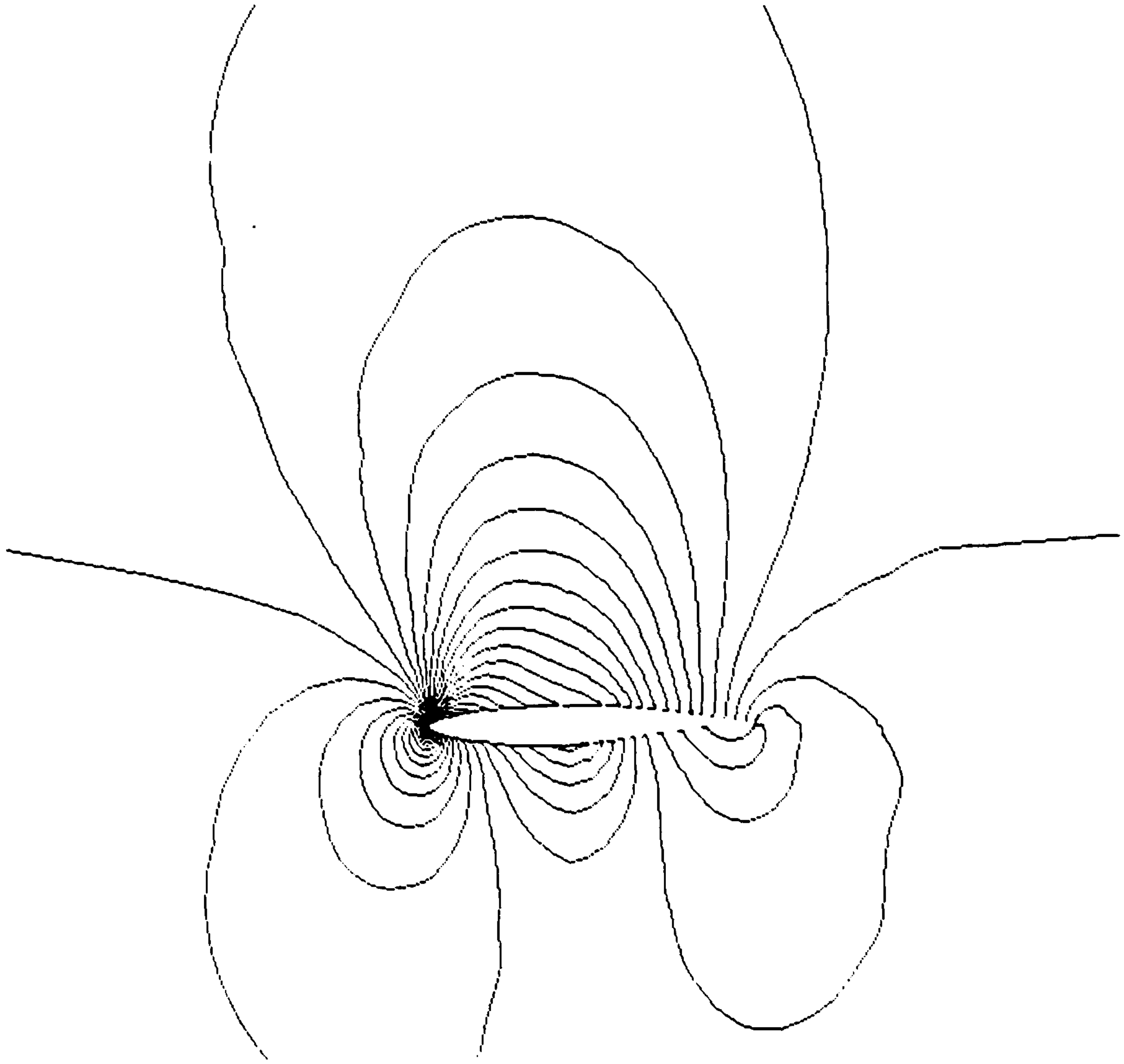
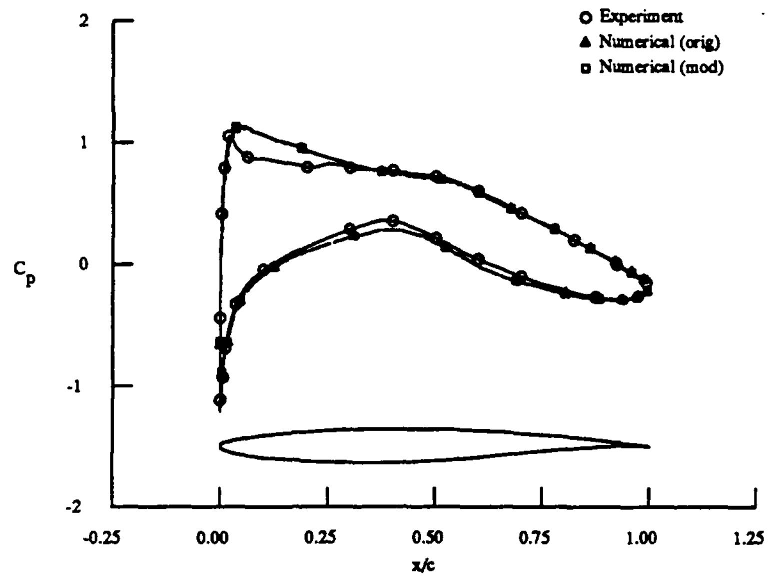


Figure 2. Grid for RAE 2822 Airfoil (128 x 32 Cells)

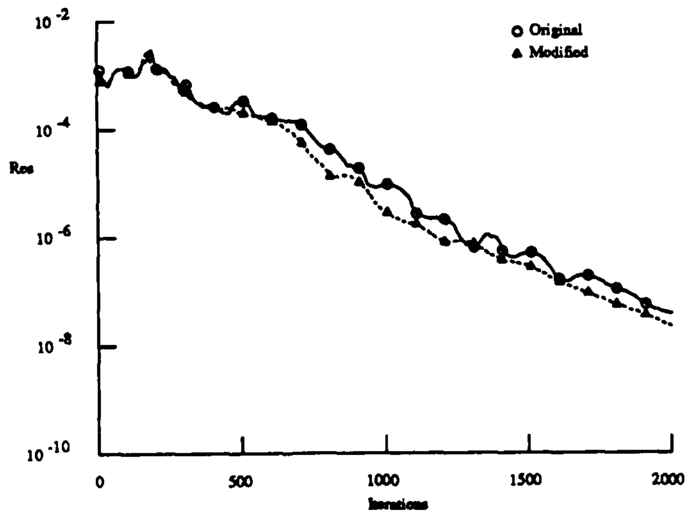


a. Static Pressure Contours

Figure 3. Results for Case 2 ($M_\infty = 0.676, \alpha = 2.40$ deg., $Re = 5.7\text{x}10^6$)

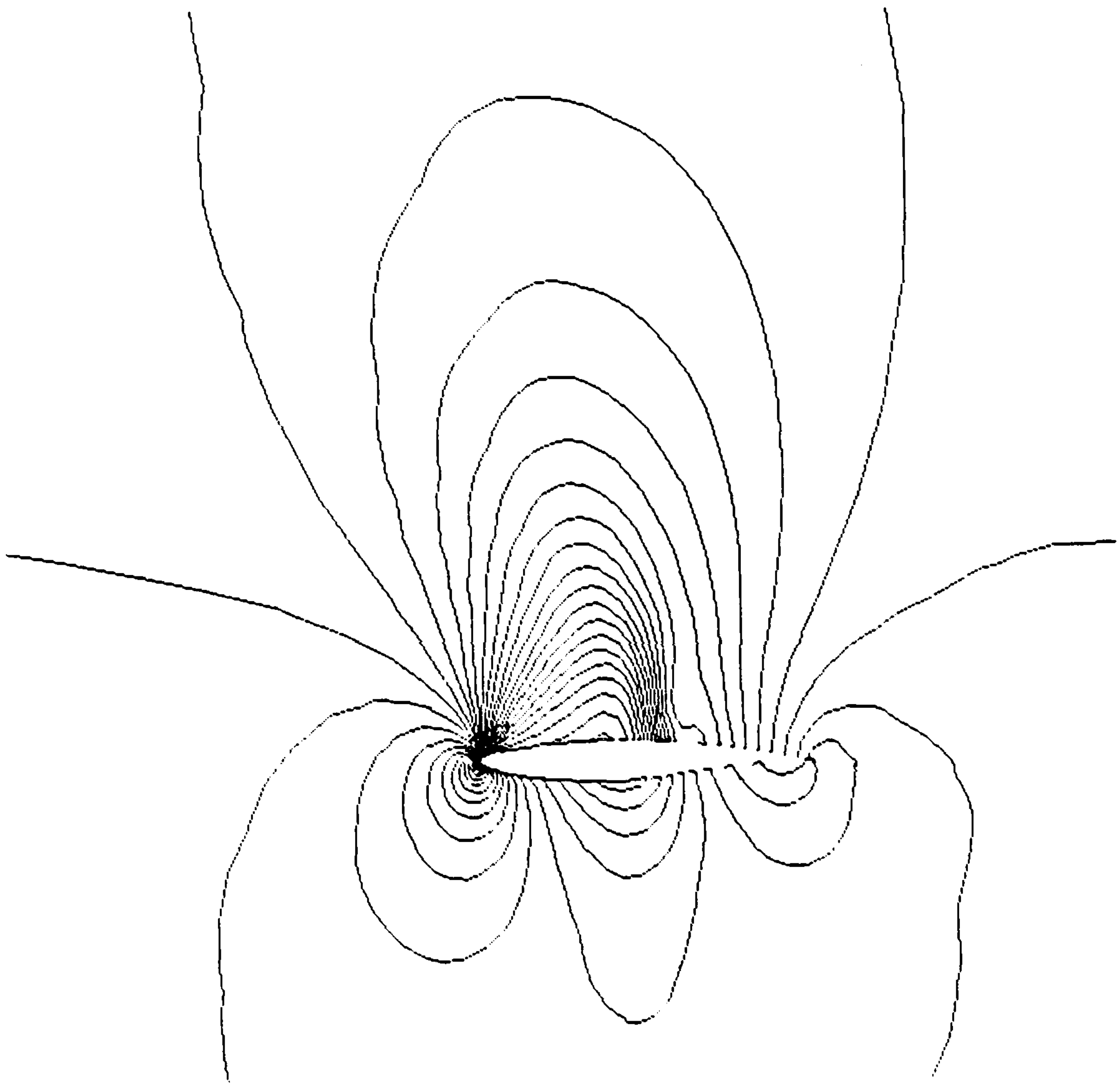


b. Pressure Coefficient



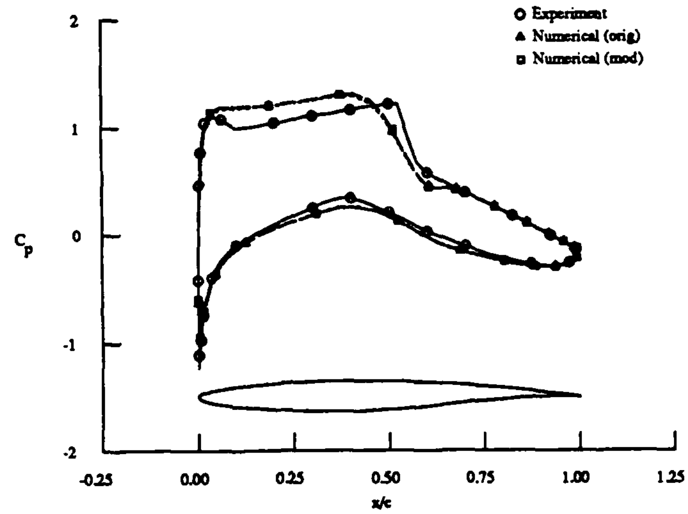
c. Convergence History

Figure 3. (Continued)

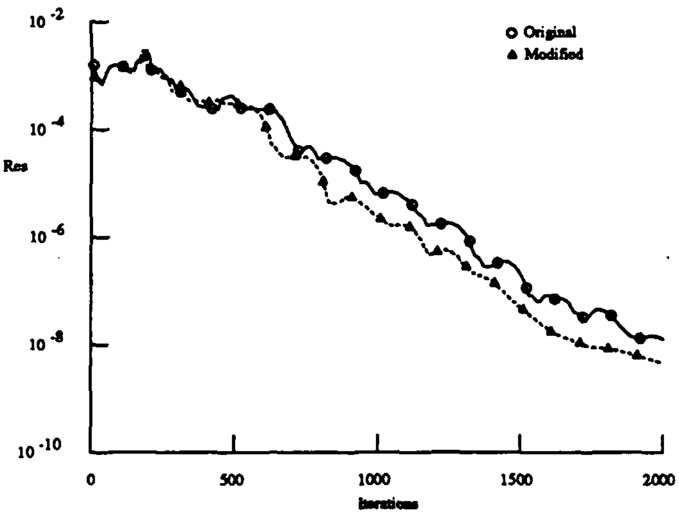


a. Static Pressure Contours

Figure 4. Results for Case 4 ($M_\infty = 0.725, \alpha = 2.92$ deg., $Re = 6.5\text{x}10^6$)

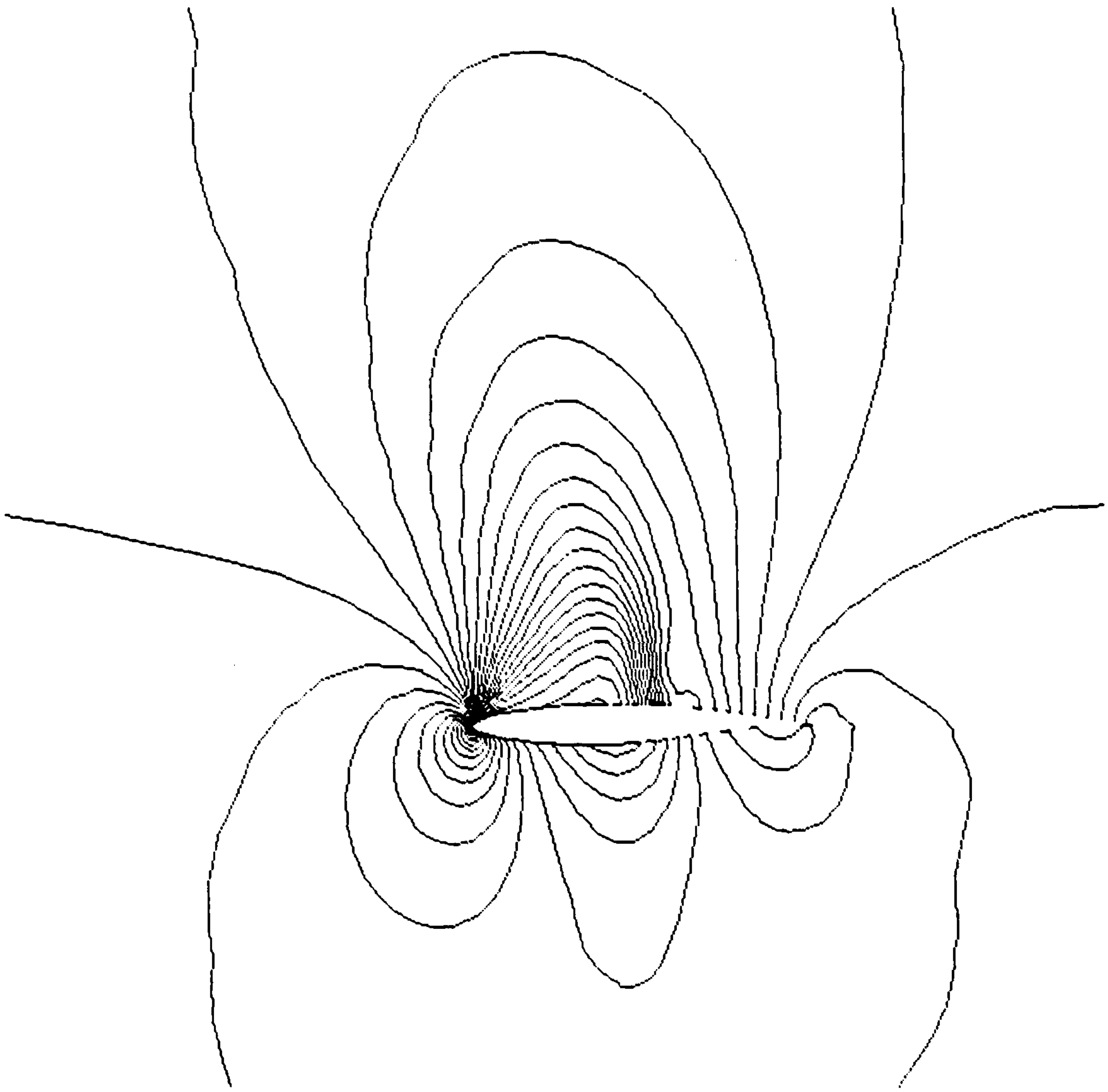


b. Pressure Coefficient



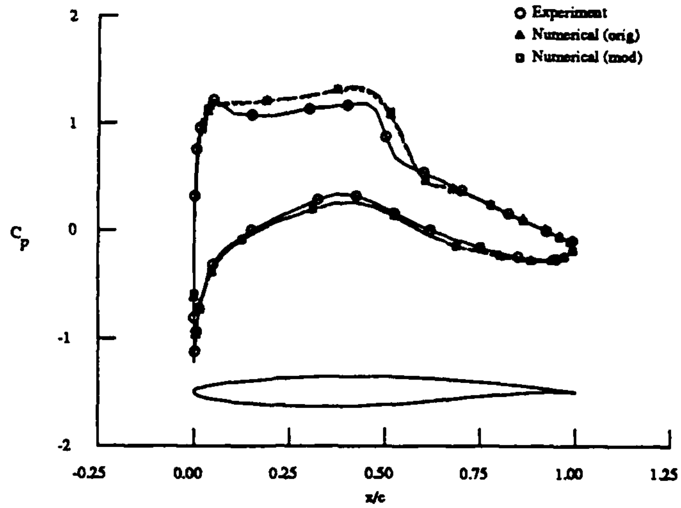
c. Convergence History

Figure 4. (Continued)

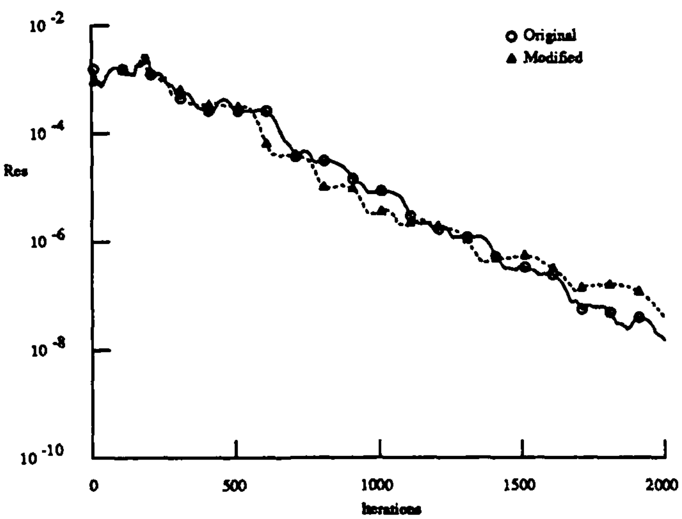


a. Static Pressure Contours

Figure 5. Results for Case 5 ($M_\infty = 0.730, \alpha = 3.19$ deg., $Re = 2.7\text{x}10^6$)



b. Pressure Coefficient



c. Convergence History

Figure 5. (Continued)

CHAPTER 5

CONCLUDING REMARKS

The locally-implicit scheme developed by Reddy and Jacocks[4] which Reddy and Nayani[5], and Nayani[6] presented for the two dimensional Navier-Stokes equations has been modified thereby eliminating the need to compute seven four-by-four coefficient matrices , of which two are stored at every node of the computational mesh. This more elegant matrix-free representation of the locally-implicit scheme results in the reduction of both, arithmetic computation and computer memory storage requirements. For a typical two-dimensional problem having 128 x 32 cells, this reduction amounts to approximately 50 percent and 30 percent for memory and arithmetic operations, respectively. An added benefit of the modification is that it simplifies the task of coding the scheme.

The multi-grid feature described and employed by Nayani[6] can be used with this scheme thereby speeding convergence in the same manner as he demonstrated with the original scheme.

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LIST OF REFERENCES

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APPENDIXES

APPENDIX 1

DERIVATION OF $C_{j,k}$

The derivation of $C_{j,k}$ is accomplished heuristically by comparison of the two-dimensional case with the one-dimensional case. In this section, that comparison will be presented to justify the terms in $C_{j,k}$. Note that $C_{j,k}$ simply multiplies $dQ_{j,k}$. Therefore, approximations in this coefficient will not affect the accuracy of the results since the terms of $Res_{j,k}$ are never tampered with.

In deriving Equation (36), Nayani[6] has shown an earlier step to be

$$\begin{aligned} CL_{j,k}\Delta Q_{j-1,k} + CR_{j,k}\Delta Q_{j+1,k} + CC_{j,k}\Delta Q_{j,k} \\ + CB_{j,k}\Delta Q_{j,k-1} + CT_{j,k}\Delta Q_{j,k+1} \\ = Res^n_{j,k} - \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}\Delta Q_{j-2,k} - \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}\Delta Q_{j+2,k} \\ - \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}\Delta Q_{j,k-2} - \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}\Delta Q_{j,k+2} \end{aligned} \tag{48}$$

For $\omega_{in} = 1$, Equation (35) becomes

$$\Delta Q^{(m+1)} = \Delta Q^{(m)} + dQ$$

Consider a sweep that is left-to-right and bottom-to-top (i.e., j increasing and k increasing).

$$\begin{aligned} CC_{j,k}dQ_{j,k} + CR_{j,k}dQ_{j+1,k} + CT_{j,k}dQ_{j,k+1} \\ + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}dQ_{j+2,k} + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}dQ_{j,k+2} = RHS \end{aligned} \tag{49}$$

$$RHS = Res^n_{j,k} - L_{j,k}(\Delta Q)$$

where $Res^n_{j,k}$ is shown in Equation (37) and $L_{j,k}(\Delta Q)$ is

$$\begin{aligned}
L_{j,k}(\Delta Q) = & CL_{j,k}\Delta Q^{(m+1)}_{j-1,k} + CB_{j,k}\Delta Q^{(m+1)}_{j,k-1} + CC_{j,k}\Delta Q^{(m)}_{j,k} \\
& + CR_{j,k}\Delta Q^{(m)}_{j+1,k} + CT_{j,k}\Delta Q^{(m)}_{j,k+1} \\
& + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}\Delta Q^{(m+1)}_{j-2,k} + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}\Delta Q^{(m)}_{j+2,k} \\
& + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}\Delta Q^{(m+1)}_{j,k-2} + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}\Delta Q^{(m)}_{j,k+2}
\end{aligned}$$

Equation (49) is analogous to the one-dimensional Equation (9)

$$du_j + \frac{c}{2}(du_{j+1}) + r(du_{j+2} - 4du_{j+1} + 6du_j) = RHS \qquad (9)$$

$$RHS = Res^n_j - L_j(\Delta u)$$

$$\begin{aligned}
L_j(\Delta u) = & \Delta u^{(m)}_j + \frac{c}{2}(\Delta u^{(m)}_{j+1} - \Delta u^{(m+1)}_{j-1}) \\
& + r(\Delta u^{(m)}_{j+2} - 4\Delta u^{(m)}_{j+1} + 6\Delta u^{(m)}_j - 4\Delta u^{(m+1)}_{j-1} + \Delta u^{(m+1)}_{j-2})
\end{aligned}$$

During the process of sweeping through the flow field, the left-hand side terms of Equation (49) represent unknowns, while the right-hand side terms represent known quantities.

The first approximation for the one-dimensional case is

$$du_{j+2} \approx du_{j+1} \approx du_j$$

Similarly, the first approximation for the two-dimensional case is

$$dQ_{j+2,k} \approx dQ_{j+1,k} \approx dQ_{j,k+2} \approx dQ_{j,k+1} \approx dQ_{j,k}$$

In one dimension, Equation (9) becomes

$$(1 + \frac{c}{2} + 3r)du_j = RHS \tag{50}$$

In two dimensions, Equation (49) becomes

$$(coef)dQ_{j,k} = RHS \tag{51}$$

where

$$coef = CC_{j,k} + CR_{j,k} + CT_{j,k} + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I$$

The second modification for the inner iteration in one dimension is to replace c by $|c|$ and for all sweeps maintain the coefficient of du_j as

$$(1 + \frac{|c|}{2} + 3r)$$

The corresponding modification in two dimensions is a series of approximations. Expanding the coefficient of $dQ_{j,k}$ in Equation (51) into all its terms gives

$$\begin{aligned} coef &= CC_{j,k} + CR_{j,k} + CT_{j,k} + \frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I \\ &= (inviscid\ terms) + (viscous\ terms) + (dissipation\ terms) \end{aligned}$$

where

$$(inviscid\ terms) = (\frac{1}{2}+\frac{1}{2})(y_\eta A^n - x_\eta B^n)_{j+\frac{1}{2},k} - \frac{1}{2}(y_\eta A^n - x_\eta B^n)_{j-\frac{1}{2},k}$$
$$+ (\frac{1}{2}+\frac{1}{2})(-y_\xi A^n + x_\xi B^n)_{j,k+\frac{1}{2}} - \frac{1}{2}(-y_\xi A^n + x_\xi B^n)_{j,k-\frac{1}{2}}$$
$$+ \frac{J^{-1}_{j,k}}{\Delta\tau} I$$

$$(viscous\ terms) = Re^{-1}[(1-1)R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}}]$$

$$(dissipation\ terms) = (1-1)\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k} I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k} I$$
$$+ (1-1)\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}} I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}} I$$
$$+ (1+3-3)\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k} I + (3-1)\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k} I$$
$$+ (1+3-3)\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}} I + (3-1)\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}} I$$

The inviscid terms at the half cell locations are approximately the same as at the cell centers. That is

$$(y_\eta A^n - x_\eta B^n)_{j+\frac{1}{2},k} \simeq (y_\eta A^n - x_\eta B^n)_{j,k}$$
$$(y_\eta A^n - x_\eta B^n)_{j-\frac{1}{2},k} \simeq (y_\eta A^n - x_\eta B^n)_{j,k}$$
$$(-y_\xi A^n + x_\xi B^n)_{j,k+\frac{1}{2}} \simeq (-y_\xi A^n + x_\xi B^n)_{j,k}$$
$$(-y_\xi A^n + x_\xi B^n)_{j,k-\frac{1}{2}} \simeq (-y_\xi A^n + x_\xi B^n)_{j,k}$$

This gives

$$(inviscid\ terms) = \frac{1}{2}(y_\eta A^n - x_\eta B^n)_{j,k} + \frac{1}{2}(-y_\xi A^n + x_\xi B^n)_{j,k} + \frac{J^{-1}_{j,k}}{\Delta\tau} I$$

The spectral radius of the first two terms of the the inviscid terms is taken next.

$$SpRad[\frac{1}{2}(y_\eta A^n - x_\eta B^n)_{j,k} + \frac{1}{2}(-y_\xi A^n + x_\xi B^n)_{j,k}]$$
$$\simeq \frac{1}{2} SpRad(y_\eta A^n - x_\eta B^n)_{j,k} + \frac{1}{2} SpRad(-y_\xi A^n + x_\xi B^n)_{j,k}$$

$$\simeq \frac{1}{2}[|y_\eta u - x_\eta v| + (x_\eta^2 + y_\eta^2)^{\frac{1}{2}} a] + \frac{1}{2}[| - y_\xi u + x_\xi v| + (x_\xi^2 + y_\xi^2)^{\frac{1}{2}} a]$$

where u and v are velocities and a is the speed of sound. CFL for the two-dimensional case is defined as

$$CFL \equiv \frac{(velocity)(\Delta time)}{(\Delta distance)} = \frac{[(velocity)(\Delta distance)](\Delta time)}{(area)}$$

where

$$[(velocity)(\Delta distance)] = [|y_\eta u - x_\eta v| + (x_\eta^2 + y_\eta^2)^{\frac{1}{2}} a] + [| - y_\xi u + x_\xi v| + (x_\xi^2 + y_\xi^2)^{\frac{1}{2}} a]$$
$$\Delta time = \Delta\tau$$
$$area = J_{j,k}^{-1}$$

At each grid node, a local time step is estimated which keeps the local Courant number approximately constant. That estimate is given by

$$\Delta\tau = \frac{(CFL)(J_{j,k}^{-1})}{[|y_\eta u - x_\eta v| + (x_\eta^2 + y_\eta^2)^{\frac{1}{2}} a] + [| - y_\xi u + x_\xi v| + (x_\xi^2 + y_\xi^2)^{\frac{1}{2}} a]}$$

The approximation to the inviscid terms with local time-stepping can be rewritten as

$$(inviscid\ terms) \simeq \frac{1}{2}CFL\frac{J^{-1}_{j,k}}{\Delta\tau}I + \frac{J^{-1}_{j,k}}{\Delta\tau}I$$

where $\Delta\tau$ is the local time step.

To make the viscous terms insensitive to sweep direction, the following approximation is made.

$$\begin{aligned}(viscous\ terms) &= Re^{-1}[(1-1)R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}}] \\ &\simeq \frac{1}{2}Re^{-1}(R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}})\end{aligned}$$

Finally, to make the dissipation terms insensitive to sweep direction, the following approximations are made.

$$(1-1)\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k}I \simeq \frac{1}{2}(\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k}I)$$

$$(1-1)\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}}I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}}I \simeq \frac{1}{2}(\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}}I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}}I)$$

$$(1+3-3)\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + (3-1)\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}I \simeq \frac{3}{2}(\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}I)$$

$$(1+3-3)\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I + (3-1)\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}I \simeq \frac{3}{2}(\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}I)$$

The resulting approximation for the dissipation terms then becomes

$$\begin{aligned}(dissipation\ terms) \simeq\ & \frac{1}{2}(\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k}I \\ & + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}}I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}}I) \\ & + \frac{3}{2}(\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}I \\ & + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}I)\end{aligned}$$

Combining the approximations for the inviscid terms, the viscous terms, and the dissipation terms gives $C_{j,k}$.

$$
\begin{aligned}
C_{j,k} = {} & CFL(\frac{J^{-1}_{j,k}}{\Delta\tau})(\frac{1}{2} + \frac{1}{CFL})I + \frac{1}{2}[\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j+\frac{1}{2},k}I \\
& + 3\frac{J^{-1}_{j+\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j+\frac{1}{2},k}I + \frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(2)}_{j-\frac{1}{2},k}I + 3\frac{J^{-1}_{j-\frac{1}{2},k}}{\Delta\tau}\epsilon^{(4)}_{j-\frac{1}{2},k}I \\
& + \frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k+\frac{1}{2}}I + 3\frac{J^{-1}_{j,k+\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k+\frac{1}{2}}I \\
& + \frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(2)}_{j,k-\frac{1}{2}}I + 3\frac{J^{-1}_{j,k-\frac{1}{2}}}{\Delta\tau}\epsilon^{(4)}_{j,k-\frac{1}{2}}I \\
& + Re^{-1}(R_{j,k+\frac{1}{2}} + R_{j,k-\frac{1}{2}})]
\end{aligned}
\tag{38}
$$

APPENDIX 2

ADDITIONAL SMOOTHING FACTORS IN SCHEME

To make the original scheme represented by Equations (34)-(36) more robust, additional smoothing factors have been added to the iteration part of the scheme. They appear in Nayani's original code, but have not been reported on in his dissertation[6]. These smoothing factors are used in the modified scheme as well. The purpose of this section is to derive these factors. Note that these factors affect only the ΔQ or dQ terms and in no way alter the $Res^n_{j,k}$ term, so accuracy of the scheme is not compromised by this addition.

The thin-layer Navier-Stokes equations in nondimensional form for generalized curvilinear coordinates was given in Section 3.1 as

$$\frac{\partial}{\partial\tau}(J^{-1}Q) + \frac{\partial}{\partial\xi}(y_\eta E - x_\eta F) + \frac{\partial}{\partial\eta}(-y_\xi E + x_\xi F) - Re^{-1}\frac{\partial}{\partial\eta}\widehat{S} = 0 \tag{33}$$

After spatially integrating Equation (33) using a finite-volume approach, an Euler implicit scheme is used. The delta form of the equation becomes

$$\frac{J^{-1}_{j,k}}{\Delta\tau}\Delta Q^n_{j,k} + [(y_\eta A^n - x_\eta B^n)\Delta Q^n]\Big|^{j+\frac{1}{2},k}_{j-\frac{1}{2},k} + [(-y_\xi A^n + x_\xi B^n)\Delta Q^n]\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} - Re^{-1}(\Delta\widehat{S}^n)\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} - \Delta d\Big|^{j+\frac{1}{2},k}_{j-\frac{1}{2},k} - \Delta d\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} = Res^n_{j,k} \tag{52}$$

where

$$A = \frac{\partial E}{\partial Q}, \quad B = \frac{\partial F}{\partial Q}$$

and $Res^n_{j,k}$ is given by Equation (37).

Consider the second term of Equation (52). It is a central difference representation for the derivative of the bracketed term.

$$[(y_\eta A^n - x_\eta B^n)\Delta Q^n]\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k} \simeq \frac{\partial}{\partial \xi}[(y_\eta A^n - x_\eta B^n)\Delta Q^n] \tag{53}$$

The derivative in Equation (53) can be split into two parts (dropping the time level n for convenience)

$$\frac{\partial}{\partial \xi}[(y_\eta A - x_\eta B)\Delta Q] = \frac{\partial}{\partial \xi}[y_\eta A \Delta Q] - \frac{\partial}{\partial \xi}[x_\eta B \Delta Q] \tag{54}$$

The first term on the right side of Equation (54) will serve as a pattern for the other derivatives. Instead of using central difference to represent $\frac{\partial}{\partial \xi}(y_\eta A \Delta Q)$ an upwind scheme is used. For an upwind scheme, A is expressed as follows. (B would be expressed similarly.)

$$A = A^+ + A^-$$
$$A^+ = \frac{A + |A|}{2}, \quad A^- = \frac{A - |A|}{2} \tag{55}$$

where $|A|$ is the spectral radius of A times the identity matrix.

$$|A| = SpRad(A)I, \quad SpRad(A) = |u| + a$$

u is velocity and a is the speed of sound. A^+ has non-negative eigenvalues and A^- has non-positive eigenvalues.

The upwind scheme uses backward difference for A^+ and forward difference for A^-. The difference form of the derivative in Equation (54) becomes

$$\frac{\partial}{\partial \xi}(y_\eta A \Delta Q) = \frac{\partial}{\partial \xi}[y_\eta (A^+ + A^-)\Delta Q]$$

$$\frac{\partial}{\partial \xi}(y_\eta A\Delta Q) \simeq (y_\eta A^+\Delta Q)_{j,k} - (y_\eta A^+\Delta Q)_{j-1,k} + (y_\eta A^-\Delta Q)_{j+1,k} - (y_\eta A^-\Delta Q)_{j,k} \tag{56}$$

Substituting Equation (55) into Equation (56) gives

$$\begin{aligned}\frac{\partial}{\partial \xi}(y_\eta A\Delta Q) &\simeq \frac{1}{2}[(y_\eta(A+|A|)\Delta Q)_{j,k} - (y_\eta(A+|A|)\Delta Q)_{j-1,k}] \\ &+ \frac{1}{2}[(y_\eta(A-|A|)\Delta Q)_{j+1,k} - (y_\eta(A-|A|)\Delta Q)_{j,k}] \\ &\simeq \frac{1}{2}[(y_\eta A\Delta Q)_{j+1,k} - (y_\eta A\Delta Q)_{j-1,k}] \\ &- \frac{1}{2}[(y_\eta|A|\Delta Q)_{j+1,k} - 2(y_\eta|A|\Delta Q)_{j,k} + (y_\eta|A|\Delta Q)_{j-1,k}] \end{aligned} \tag{57}$$

The first bracketed term of Equation (57) is simply the central difference representation of $\frac{\partial}{\partial \xi}(y_\eta A\Delta Q)$, that is

$$\frac{1}{2}[(y_\eta A\Delta Q)_{j+1,k} - (y_\eta A\Delta Q)_{j-1,k}] = (y_\eta A\Delta Q)\Big|_{j-\frac{1}{2},k}^{j+\frac{1}{2},k}$$

The second bracketed term of Equation (57) is the additional smoothing that results from using upwind differencing on the ΔQ side of Equation (52). It is the central difference representation for the second derivative for $(y_\eta A\Delta Q)$, that is

$$-\frac{1}{2}[(y_\eta|A|\Delta Q)_{j+1,k} - 2(y_\eta|A|\Delta Q)_{j,k} + (y_\eta|A|\Delta Q)_{j-1,k}] = -\frac{1}{2}\delta_\xi^2(y_\eta|A|\Delta Q)_{j,k}$$

where δ_ξ^2 is the central difference operator representing $\frac{\partial^2}{\partial \xi^2}$.

The pattern is the same for $\frac{\partial}{\partial \xi}(x_\eta B\Delta Q)$, $\frac{\partial}{\partial \eta}(-y_\xi A\Delta Q)$, and $\frac{\partial}{\partial \eta}(x_\xi B\Delta Q)$. When upwind differencing is accomplished for all of these terms, then Equation (52) becomes

$$
\begin{aligned}
&\frac{J^{-1}_{j,k}}{\Delta\tau}\Delta Q^n_{j,k} + \left[(y_\eta A^n - x_\eta B^n)\Delta Q^n\right]\Big|^{j+\frac{1}{2},k}_{j-\frac{1}{2},k} + \left[(-y_\xi A^n + x_\xi B^n)\Delta Q^n\right]\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} \\
&\qquad - (fac)\{\delta^2_\xi\left[(y_\eta|A^n| - x_\eta|B^n|)\Delta Q^n\right]_{j,k} + \delta^2_\eta\left[(-y_\xi|A^n| + x_\xi|B^n|)\Delta Q^n\right]_{j,k}\} \\
&\qquad - Re^{-1}(\Delta\widehat{S}^n)\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} - \Delta d\Big|^{j+\frac{1}{2},k}_{j-\frac{1}{2},k} - \Delta d\Big|^{j,k+\frac{1}{2}}_{j,k-\frac{1}{2}} = Res^n_{j,k} \qquad (58)
\end{aligned}
$$

The (fac) coefficient before the second differences gives control of the amount of smoothing that is used. A typical value is $(fac) = 0.1$. Equation (58) reflects the equation that Nayani's code[6] actually solves, not Equation (52). In the present modified code also, implicit smoothing terms are used in the iteration process with $(fac) = 0.1$.

VITA

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